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Percolation points and critical point in the Ising model

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Abstract. Rigorous inequalities are proved, which relate percolation probability, mean cluster size and pair connectedness respectively with magnetization, susceptibility and pair correlation function in ferromagnetic Ising models. In two dimensions the critical point is shown to be a percolation point, while in three dimensions this is not true.

1. Introduction

The site and bond percolation problem has been applied to various phenomena (for review articles see Frisch and Hammersley 1963, Shante and Kirkpatrick 1971, Essam 1973). For a given lattice and a given density of particles distributed at random on its sites, one can look at the distribution of clusters of particles linked together by nearest-neighbour bonds. The main task of the site percolation problem is to evaluate: (i) the percolation probability, i.e. the probability that a given particle belongs to an infinite cluster; (ii) the mean cluster size, which is a weighted average of the cluster size; (iii) the pair connectedness, which is defined as the probability that two given sites are connected by at least one chain of occupied sites belonging to finite clusters. There is a critical density where these quantities behave in a very similar way to spontaneous magnetization, the susceptibility, and the pair correlation function in a ferromagnet near the critical point.

A general model has been introduced by Kasteleyn and Fortuin (1969, 1972), which includes as a particular case the bond percolation problem and the Ising model. With this formulation it has been possible to apply the usual Hamiltonian technique, including the renormalization group, to the percolation problem (Harris *et al* 1975, Amit 1976, Priest and Lubensky 1976).

The existence of an infinite cluster plays an important role in the theory of dilute systems such as ferromagnets (Essam 1973, Sato *et al* 1959, Elliott and Heap 1962, Rushbrooke *et al* 1972, Rapaport 1972a, b, Stauffer 1975a, b, Griffiths and Lebowitz 1968, Bishop 1975, Young 1976) and inhomogeneous conductors (Kirkpatrick 1973 and references cited therein, Stinchcombe 1974). Some of these problems have been treated with the renormalization group approach (Lubensky 1975, Krey 1975a, b, Harris *et al* 1976).

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A suggestion has been advanced by Bishop (1973, 1974) to relate the Curie temperature T_C to the critical probability of the bond and the site percolation problem.

By introducing a ferromagnetic interaction among the particles, the system changes into a lattice gas model, or, in the magnetic terminology, into a Ising model. In such a model, one can study at the same time percolation and phase transitions in order to find out if there is an influence of one over the other. Since percolation also occurs with zero interaction, while phase transition does not, one should expect, if any connection exists between them, that the former would influence the latter.

The cluster distribution in an Ising model has been considered by Binder (1975) and Müller-Krumbhaar (1974), Binder *et al* (1976) in connection with the nucleation problem, and by Domb (1974) who suggested that only ramified clusters play a significant role in the neighbourhood of the critical point. This suggestion has been checked later by means of Monte Carlo calculations (Domb *et al* 1975). We shall discuss this point in the last section. Recently Lebowitz (1976) has derived some cluster inequalities in the Ising model, which have been found useful in the study of metastable states.

Kikuchi (1970) first derived an approximation for the percolation problem with interaction. The percolation probability in a three-dimensional ferromagnetic Ising model has been calculated by means of Monte Carlo technique by Müller-Krumbhaar (1974b), who concluded that for zero external field, an infinite cluster of overturned spins appears at a temperature T_p below the Curie temperature T_C. The same result was found on a large class of branching media such as Bethe lattices, by Coniglio (1976) who has also suggested (Coniglio 1975) that this should be the case for any three-dimensional system, while for two-dimensional systems T_p and T_C should coincide. This last conjecture was advanced on a different basis by Odagaki *et al* (1975).

Recently, in a previous paper (Coniglio *et al* 1976, to be referred to as I), we have studied the connection between phase transitions and percolation in an Ising model in a rigorous way, proving that such a link exists. The main result was to prove two theorems: (1) in a ferromagnetic Ising model, the spontaneous magnetization is less than or equal to the percolation probability, and (2) for a large class of two-dimensional models with ferromagnetic interaction there is no coexistence of infinite clusters of spins 'up' and spins 'down'. This is a generalization of a theorem proved by Harris (1960) for the random bond problem on the square lattice. Fisher (1961) subsequently proved it for the random site problem on a large class of planar lattices. Recently Miyamoto (1975) has generalized Harris's result to a class of interacting systems.

In this paper we shall continue the study of the connection between phase transitions and percolation in the Ising model. In § 2 we first introduce the terminology and then, from general properties of the Ising model, we obtain some inequalities from which we derive theorem 1 of I. Other relations between correlation function and pair connectedness, which lead to an inequality between the susceptibility and the mean cluster size are derived in § 3.

As a consequence of theorems 1 and 2, in § 4 it will be shown that in a two-dimensional Ising model at $H=0$, the critical temperature T_c and the percolation temperature T_p coincide. This, combined with the result of § 2 leads to inequalities between 'percolative' critical indices and the usual ferromagnetic critical indices β, γ, ν (Fisher 1967a).

In § 5 a phase diagram is suggested which shows distribution of infinite clusters in the density–temperature and external field–temperature plane. The conclusions follow in § 6.

2. Spontaneous magnetization and infinite clusters

Consider a d-dimensional Ising model with nearest-neighbour ferromagnetic interaction. Let Z^d be the d-dimensional lattice whose points have integer coordinates in R^d. Introduce the lattice gas variables relative to the point of coordinate i:

$$\pi_i = \tfrac{1}{2}(1-\sigma_i); \qquad \tilde{\pi}_i = \tfrac{1}{2}(1+\sigma_i);$$

where $\sigma_i = \pm 1$ is the usual spin variable.

For a finite set $A \subset Z^d$ we define $\pi^A = \prod_{i\in A} \pi_i$; $\tilde{\pi}^A = \prod_{i\in A} \tilde{\pi}_i$.

In a given configuration a (+)-cluster ((−)-cluster) is defined as the maximum number of 'up' ('down') spins such that any two of them can be connected by at least one chain of nearest-neighbour 'up' ('down') spins.

Let us call $\gamma_i(\gamma_i^\infty)$ the characteristic functions of the event that the spin in i belongs to a finite (infinite) (−)-cluster and $\gamma_{ij}(\gamma_{ij}^\infty)$ the characteristic function of the event that i and j belong to the same finite (infinite) (−)-cluster.

$\tilde{\gamma}_i$, $\tilde{\gamma}_i^\infty$, $\tilde{\gamma}_{ij}$ and $\tilde{\gamma}_{ij}^\infty$ will denote the analogous functions for (+)-clusters. Of course, the following relations hold:

$$\pi_i = \gamma_i + \gamma_i^\infty; \qquad \tilde{\pi}_i = \tilde{\gamma}_i + \tilde{\gamma}_i^\infty. \tag{1}$$

A function $F(\boldsymbol{\sigma})$ of the variable $\boldsymbol{\sigma} = \{\sigma_1, \ldots, \sigma_N\}$ is said to be non-decreasing if $F(\sigma_1, \ldots, \sigma_N) \leqslant F(\sigma_1', \ldots, \sigma_N')$ when $\sigma_i \leqslant \sigma_i'$ and non-increasing in the opposite case.

If Λ_0 is a d-dimensional cube containing the origin, $\partial\Lambda_0$ its boundary, and $\boldsymbol{\sigma} \subset \Lambda_0$, we denote by $\langle F(\boldsymbol{\sigma})\rangle_{\Lambda_\pm}$ the thermal average for the Ising model in the volume $\Lambda = \Lambda_0 \cup \partial\Lambda_0$ with (±)-boundary conditions on $\partial\Lambda_0$ while $\langle F(\boldsymbol{\sigma})\rangle_\Lambda$ denotes the thermal average with zero boundary conditions.

By definition:

$$\langle F(\boldsymbol{\sigma})\rangle_{\Lambda_+} = \frac{\langle F(\boldsymbol{\sigma})\tilde{\pi}^{\partial\Lambda_0}\rangle_\Lambda}{\langle \tilde{\pi}^{\partial\Lambda_0}\rangle_\Lambda}; \qquad \langle F(\boldsymbol{\sigma})\rangle_{\Lambda_-} = \frac{\langle F(\boldsymbol{\sigma})\pi^{\partial\Lambda_0}\rangle_\Lambda}{\langle \pi^{\partial\Lambda_0}\rangle_\Lambda}.$$

Moreover, we define:

$$\langle F(\boldsymbol{\sigma})\rangle_\pm = \lim_{\Lambda_0\to\infty} \langle F(\boldsymbol{\sigma})\rangle_{\Lambda_\pm}; \qquad \langle F(\boldsymbol{\sigma})\rangle = \lim_{\Lambda_0\to\infty} \langle F(\boldsymbol{\sigma})\rangle_\Lambda.$$

We recall here two properties of the ferromagnetic nearest-neighbour Ising model.

(i) FKG inequality (Fortuin *et al* 1971). If $f(\boldsymbol{\sigma})$ and $g(\boldsymbol{\sigma})$ are both non-decreasing or non-increasing functions of $\boldsymbol{\sigma}$, then $\langle f(\boldsymbol{\sigma})g(\boldsymbol{\sigma})\rangle_\Lambda \geqslant \langle f(\boldsymbol{\sigma})\rangle_\Lambda \langle g(\boldsymbol{\sigma})\rangle_\Lambda$. Consequently, if either $f(\boldsymbol{\sigma})$ or $g(\boldsymbol{\sigma})$ is non-increasing when the other is non-decreasing, we have $\langle f(\boldsymbol{\sigma})g(\boldsymbol{\sigma})\rangle_\Lambda \leqslant \langle f(\boldsymbol{\sigma})\rangle_\Lambda \langle g(\boldsymbol{\sigma})\rangle_\Lambda$. As an example we have:

$$\langle \pi^A \pi^B\rangle \geqslant \langle \pi^A\rangle\langle \pi^B\rangle; \tag{2}$$

$$\langle \pi^A \tilde{\pi}^B\rangle \leqslant \langle \pi^A\rangle\langle \tilde{\pi}^B\rangle; \tag{3}$$

for any finite set A and $B \subset Z^d$.

(ii) Markov property. Given a set $A \subset Z^d$ and its boundary ∂A, let us denote by $P(Y|X \cup \partial X)$ the probability of an event Y outside A given the configuration $X \cup \partial X$ on $A \cup \partial A$. By definition:

$$P(Y|X \cup \partial X) = P(Y, X \cup \partial X)/P(X \cup \partial X),$$

where $P(Y, X \cup \partial X)$ is the probability that both the events Y and $X \cup \partial X$ occur. The

Markovian property states that $P(Y|X \cup \partial X) = P(Y|\partial X)$. As a consequence of properties (i) and (ii) we shall derive the following inequalities which will be used later on:

$$\langle \pi^A \tilde{\pi}^{\partial A} \chi_B \rangle_\Lambda \geq \langle \pi^A \tilde{\pi}^{\partial A} \rangle_\Lambda \langle \chi_B \rangle_\Lambda; \tag{4}$$

$$\langle \tilde{\pi}^A \pi^{\partial A} \chi_B \rangle_\Lambda \leq \langle \tilde{\pi}^A \pi^{\partial A} \rangle_\Lambda \langle \chi_B \rangle_\Lambda; \tag{5}$$

where $A \subset Z^d$ and χ_B is the non-decreasing characteristic function of an event B outside A.

In fact, by applying first the Markovian property and then the FKG inequality, we have:

$$\langle \pi^A \tilde{\pi}^{\partial A} \chi_B \rangle_\Lambda = \frac{\langle \pi^A \tilde{\pi}^{\partial A} \chi_B \rangle_\Lambda}{\langle \pi^A \tilde{\pi}^{\partial A} \rangle_\Lambda} \langle \pi^A \tilde{\pi}^{\partial A} \rangle_\Lambda = \frac{\langle \tilde{\pi}^{\partial A} \chi_B \rangle_\Lambda}{\langle \tilde{\pi}^{\partial A} \rangle_\Lambda} \langle \pi^A \pi^{\partial A} \rangle_\Lambda \geq \langle \chi_B \rangle_\Lambda \langle \pi^A \tilde{\pi}^{\partial A} \rangle_\Lambda; \tag{6}$$

$$\langle \tilde{\pi}^A \pi^{\partial A} \chi_B \rangle_\Lambda = \frac{\langle \tilde{\pi}^A \pi^{\partial A} \chi_B \rangle_\Lambda}{\langle \tilde{\pi}^A \pi^{\partial A} \rangle_\Lambda} \langle \tilde{\pi}^A \pi^{\partial A} \rangle_\Lambda = \frac{\langle \pi^{\partial A} \chi_B \rangle_\Lambda}{\langle \pi^{\partial A} \rangle_\Lambda} \langle \tilde{\pi}^A \pi^{\partial A} \rangle_\Lambda \leq \langle \chi_B \rangle_\Lambda \langle \tilde{\pi}^A \pi^{\partial A} \rangle_\Lambda. \tag{7}$$

Of course (6) and (7) hold in the opposite direction ($\leq \leftrightarrow \geq$) if χ_B is non-increasing.

Now, let us prove the following.

Theorem 1. In a ferromagnetic Ising model with nearest-neighbour interaction for zero external field ($H = 0$) and below the critical temperature ($T < T_c$) the following inequality holds:

$$M(0^+, T) \leq R_\uparrow(0^+, T) - R_\downarrow(0^+, T); \tag{8}$$

where $M(0^+, T)$ is the reduced spontaneous magnetization and $R_\uparrow(0^+, T)$ ($R_\downarrow(0^+, T)$) is the density of 'up' ('down') spins belonging to an infinite (+)-cluster ((−)-cluster) in the positive phase ((+)-boundary condition). The percolation probability for 'up' ('down') spins is obtained by dividing $R_\uparrow(R_\downarrow)$ by the density of 'up' ('down') spins.

With the symbol $0^+(0^-)$ we shall always mean $H = 0$ and (+)-boundary ((−)-boundary) condition.

Proof. We have:

$$M(0^+, T) = \langle \tilde{\pi}_0 \rangle_+ - \langle \pi_0 \rangle_+ = \langle \tilde{\gamma}_0 \rangle_+ - \langle \gamma_0 \rangle_+ + \langle \tilde{\gamma}_0^\infty \rangle_+ - \langle \gamma_0^\infty \rangle_+. \tag{9}$$

The subscript 0 refers to the origin. Since by definition

$$\langle \tilde{\gamma}_0^\infty \rangle_+ = R_\uparrow(0^+, T); \qquad \langle \gamma_0^\infty \rangle_+ = R_\downarrow(0^+, T); \tag{10}$$

we will show that

$$\langle \tilde{\gamma}_0 \rangle_+ - \langle \gamma_0 \rangle_+ \leq 0 \tag{11}$$

in order to prove the theorem.

The above inequality expresses the fact that in the positive phase the density of 'up' spins belonging to finite clusters is less than or equal to the corresponding density of 'down' spins, even though the total density of 'up' spins is larger than the total density of 'down' spins.

By definition:

$$\langle \tilde{\gamma}_0 \rangle_+ = \sum_s \tilde{n}_{0s}(0^+, T); \qquad \langle \gamma_0 \rangle_+ = \sum_s n_{0s}(0^+, T); \tag{12}$$

where $\tilde{n}_{0s}(0^+, T)$ ($n_{0s}(0^+, T)$) is the mean number of (+)-clusters ((−)-clusters) of s spins, containing the origin 0 in the positive phase, and is given by

$$\tilde{n}_{0s}(0^+, T) = \sum_{\{A_{0s}\}} \lim_{\Lambda\to\infty} \frac{\langle \tilde{\pi}^{A_{0s}} \pi^{\partial A_{0s}} \tilde{\pi}^{\partial\Lambda_0} \rangle_\Lambda}{\langle \tilde{\pi}^{\partial\Lambda_0} \rangle_\Lambda}; \tag{13}$$

$$n_{0s}(0^+, T) = \sum_{\{A_{0s}\}} \lim_{\Lambda\to\infty} \frac{\langle \pi^{A_{0s}} \tilde{\pi}^{\partial A_{0s}} \tilde{\pi}^{\partial\Lambda_0} \rangle_\Lambda}{\langle \tilde{\pi}^{\partial\Lambda_0} \rangle_\Lambda}; \tag{14}$$

where A_{0s} is a cluster of s spins containing the origin and ∂A_{0s} is its boundary. The sum is over all the finite clusters of s spins containing the origin. From (6) and (7):

$$\frac{\langle \tilde{\pi}^{A_{0s}} \pi^{\partial A_{0s}} \tilde{\pi}^{\partial\Lambda_0} \rangle_\Lambda}{\langle \tilde{\pi}^{\partial\Lambda} \rangle_\Lambda} \leqslant \langle \tilde{\pi}^{A_{0s}} \pi^{\partial A_{0s}} \rangle_\Lambda; \tag{15}$$

$$\frac{\langle \pi^{A_{0s}} \tilde{\pi}^{\partial A_{0s}} \tilde{\pi}^{\partial\Lambda_0} \rangle_\Lambda}{\langle \tilde{\pi}^{\partial\Lambda_0} \rangle_\Lambda} \geqslant \langle \pi^{A_{0s}} \tilde{\pi}^{\partial A_{0s}} \rangle. \tag{16}$$

For the symmetry ($H = 0$):

$$\langle \tilde{\pi}^{A_{0s}} \pi^{\partial A_{0s}} \rangle_\Lambda = \langle \pi^{A_{0s}} \tilde{\pi}^{\partial A_{0s}} \rangle_\Lambda. \tag{17}$$

Consequently, from (13)–(16):

$$\tilde{n}_{0s}(0^+, T) \leqslant n_{0s}(0^+, T). \tag{18}$$

Equation (11) follows from (12) which proves the theorem.

This theorem says that spontaneous magnetization can only exist if there is an infinite cluster. The reverse statement is not true. This is obvious in a pure percolation problem without interaction, where numerical calculations (Sykes *et al* 1976a, see also Shante and Kirkpatrick 1971, Essam 1973) in three-dimensional models show that the critical concentration is less than 1/2, which means that an infinite cluster can well exist without magnetization.

This theorem also does not say that the magnetization always implies (including $H \neq 0$) an infinite cluster, as numerical calculations (Sykes *et al* 1976b) show in two-dimensional models at zero interaction, where the critical concentration is found to be larger than 1/2. In conclusion, what we learn from this theorem is that the infinite cluster is the necessary structure which can carry the information of the long-range order.

3. Spin–spin correlation function and pair connectedness

Before we prove the second theorem let us give a few definitions.

The pair correlation function is defined as

$$g_{ij} = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle; \tag{19}$$

or in terms of lattice gas variables:

$$g_{ij} = 4(\langle \pi_i \pi_j \rangle - \langle \pi_i \rangle \langle \pi_j \rangle); \tag{20}$$

$$g_{ij} = 4(\langle \tilde{\pi}_i \tilde{\pi}_j \rangle - \langle \tilde{\pi}_i \rangle \langle \tilde{\pi}_j \rangle). \tag{21}$$

Let us also define the pair correlation function for spins belonging to infinite clusters:

$$g_{ij}^{\infty}=4(\langle\gamma_i^{\infty}\gamma_j^{\infty}\rangle-\langle\gamma_i^{\infty}\rangle\langle\gamma_j^{\infty}\rangle), \qquad \text{for 'down' spins;} \tag{22}$$

$$\tilde{g}_{ij}^{\infty}=4(\langle\tilde{\gamma}_i^{\infty}\tilde{\gamma}_j^{\infty}\rangle-\langle\tilde{\gamma}_i^{\infty}\rangle\langle\tilde{\gamma}_j^{\infty}\rangle), \qquad \text{for 'up' spins.} \tag{23}$$

The pair connectedness function is defined as the probability that two spins i and j belong to the same finite cluster and is given by

$$p_{ij}=\langle\gamma_{ij}\rangle, \qquad \text{for 'down' spins;} \tag{24}$$

$$\tilde{p}_{ij}=\langle\tilde{\gamma}_{ij}\rangle, \qquad \text{for 'up' spins.} \tag{25}$$

Note that $g_{ij}=\tilde{g}_{ij}$ and $p_{ij}=\tilde{p}_{ij}$ only in the symmetrical case ($H=0$, $T>T_c$) while (20) and (21) are always equal even though they are the correlation functions for 'up' and 'down' spins.

Now, we shall prove the following theorem.

Theorem 2. For a ferromagnetic Ising model with nearest-neighbour interaction we have:

$$g_{ij}\leqslant 4p_{ij}+g_{ij}^{\infty}; \tag{26}$$

$$g_{ij}\leqslant 4\tilde{p}_{ij}+\tilde{g}_{ij}^{\infty}. \tag{27}$$

Proof. From (1):

$$\langle\pi_i\pi_j\rangle=\langle\gamma_i\pi_j\rangle+\langle\gamma_i^{\infty}\pi_j\rangle=p_{ij}+\sum_{\substack{A\ni i\\ A\not\ni j}}\langle\pi^A\tilde{\pi}^{\partial A}\pi_j\rangle+\langle\gamma_i^{\infty}\gamma_j^{\infty}\rangle+\sum_{\substack{B\not\ni i\\ B\ni j}}\langle\gamma_i^{\infty}\pi^B\tilde{\pi}^{\partial B}\rangle;$$

where A and B are clusters of s spins.

π_j and γ_i^{∞} are non-increasing functions, so that we can apply inequality (6) in the opposite direction:

$$\langle\pi_i\pi_j\rangle\leqslant p_{ij}+\sum_{\substack{A\ni i\\ A\not\ni j}}\langle\pi^A\tilde{\pi}^{\partial A}\rangle\langle\pi_j\rangle+\langle\gamma_i^{\infty}\gamma_j^{\infty}\rangle+\sum_{\substack{B\not\ni i\\ B\ni j}}\langle\gamma_i^{\infty}\rangle\langle\pi^B\tilde{\pi}^{\partial B}\rangle.$$

By adding and subtracting $\langle\gamma_i^{\infty}\rangle\langle\gamma_j^{\infty}\rangle$ we have:

$$\langle\pi_i\pi_j\rangle-\langle\pi_i\rangle\langle\pi_j\rangle\leqslant p_{ij}+\tfrac{1}{4}g_{ij}^{\infty};$$

and, from (20):

$$g_{ij}\leqslant 4p_{ij}+g_{ij}^{\infty}. \tag{28}$$

The second part, equation (27), is proved in the same way.

It is not difficult to prove also that

$$\tfrac{1}{4}g_{ij}^{\infty}\leqslant\langle\gamma_{ij}^{\infty}\rangle\equiv p_{ij}^{\infty}; \qquad \tfrac{1}{4}\tilde{g}_{ij}^{\infty}\leqslant\langle\tilde{\gamma}_{ij}^{\infty}\rangle\equiv\tilde{p}_{ij}^{\infty};$$

which, together with (26) and (27), give:

$$\tfrac{1}{4}g_{ij}\leqslant p_{ij}+p_{ij}^{\infty}; \qquad \tfrac{1}{4}g_{ij}\leqslant\tilde{p}_{ij}+\tilde{p}_{ij}^{\infty}; \tag{29}$$

which is analogous to theorem 1.

Inequality (29) states that long-range correlation exists if there is large connectedness. In other words the information can propagate from i to j only if i and j are connected by an 'open' path. This was conjectured by Reatto and Rastelli (1972), who produced on this basis, an explicit form for the spin–spin correlation function.

From theorem 2 we can derive some relations between the susceptibility and the mean cluster size. The susceptibility satisfies the following relation (Fisher 1967a):

$$\chi(H, T) = \sum_j g_{0j}(H, T).$$

By analogy define

$$\chi_\downarrow^\infty(H, T) = \sum_j g_{0j}^\infty(H, T); \qquad \chi_\uparrow^\infty(H, T) = \sum_j \tilde{g}_{0j}^\infty(H, T).$$

The mean cluster size of 'down' spins is defined by

$$S_\downarrow(H, T) = \sum_s sn_{0s}(H, T)\left(\sum_s n_{0s}(H, T)\right)^{-1}; \tag{30}$$

and is related to the pair connectedness function (Essam 1973) according to the following relation:

$$S_\downarrow(H, T) = \frac{\sum_j p_{0j}(H, T)}{\langle\gamma_0\rangle}.$$

The same relations hold for $S_\uparrow(H, T)$ and $\tilde{p}_{0j}$.

From theorem 2 we have:

$$\chi \leqslant 4\langle\gamma_0\rangle S_\downarrow + \chi_\downarrow^\infty, \qquad \chi \leqslant 4\langle\tilde{\gamma}_0\rangle S_\uparrow + \chi_\uparrow^\infty. \tag{31}$$

We will show in the next section that all these relations, valid for all dimensionalities, can be specialized in two dimensions by using the following theorem proved in I.

Theorem 3. For a square ferromagnetic Ising model with nearest-neighbour interaction we have:

$$R_\uparrow(H, T)R_\downarrow(H, T) = 0. \tag{32}$$

This equation states that an infinite (+)-cluster can never coexist with an infinite (−)-cluster. This is an old result proved by Harris (1960) for the random bond problem, and generalized by Fisher (1961) to the site problem on a large class of planar lattices. Recently Miyamoto (1975) extended Harris's result to the bond problem with ferromagnetic interaction for the symmetrical case ($H = 0$, $T \geqslant T_c$). Theorem 3 can also be extended to all simple planar graphs which admit an elementary cell and two axes of symmetry.

4. Two-dimensional model: coincidence of critical point with percolation point

In this section we consider two-dimensional ferromagnetic Ising models with nearest-neighbour interaction for which theorems 1, 2 and 3 hold.

The density of 'down' spins is given by

$$p(H, T) = \langle\pi_0\rangle. \tag{33}$$

Define for $T \geqslant T_c$:

$$p_c(T) = \sup_{T_{\text{fixed}}} \{p(H, T) : R_\downarrow(H, T) = 0\} \tag{34}$$

where $p_c(\infty) = p_c^0$ is the critical probability in the random case.

For the symmetry:

$$R_\uparrow(H, T) = R_\downarrow(-H, T). \tag{35}$$

From theorem 3, for $H = 0$ and $T \geqslant T_c$:

$$R_\downarrow(0, T) = 0; \tag{36}$$

and from (34), $p_c(T) \geqslant \frac{1}{2}$.

For $T < T_c$ definition (34) may be misleading, since the density does not change continuously across the coexistence curve. Therefore (34) can be extended to the values $T_p \leqslant T < T_c$ for which $p(0^+, T_p) = p_c(T_p) \leqslant p_c(T) \leqslant p(0^+, T)$.

It is also convenient to define critical probabilities on the coexistence curve ($H = 0$, $T < T_c$). The density of overturned spins along such a curve is given by:

$$p = p(T) = \begin{cases} \langle \pi_0 \rangle_+, & 0 \leqslant T \leqslant T_c, 0 \leqslant p \leqslant \frac{1}{2}; \\ \langle \pi_0 \rangle_-, & T_c > T \geqslant 0, \frac{1}{2} < p \leqslant 1. \end{cases} \tag{37}$$

Express $R_\downarrow$ as function of p:

$$R_\downarrow(p) = \begin{cases} R_\downarrow(0^+, T), & 0 \leqslant T \leqslant T_c, 0 \leqslant p \leqslant \frac{1}{2}; \\ R_\downarrow(0^-, T), & T_c > T \geqslant 0, \frac{1}{2} < p \leqslant 1. \end{cases} \tag{38}$$

The critical density on the coexistence curve is given by

$$p_c = \sup\{p : R_\downarrow(p) = 0\}. \tag{39}$$

From theorem 1 and (35):

$$\left.\begin{aligned} &R_\uparrow(0^+, T) = R_\downarrow(0^-, T) \geqslant M(0^+, T) > 0 \\ &R_\downarrow(0^+, T) = 0 \end{aligned}\right\} \quad (H = 0^+, T < T_c) \tag{40}$$

and

$$R_\downarrow(0, T) = R_\uparrow(0, T) = 0 \qquad (H = 0, T \geqslant T_c). \tag{41}$$

From (38)–(40) along the coexistence curve:

$$p_c = \tfrac{1}{2}. \tag{42}$$

In other words, for $H = 0$, the critical point and percolation point coincide for any sufficiently regular planar lattice with nearest-neighbour ferromagnetic interaction.

From (40) and (41) $R_\uparrow(0^+, T)$ is different from zero for $T < T_c$ and zero for $T \geqslant T_c$. We do not know if there is a jump or not at T_c.

If we suppose that

$$\lim_{T \to T_c} \langle \tilde{\gamma}_0 \rangle_+ = \lim_{T \to T_c} \langle \gamma_0 \rangle_+, \tag{43}$$

where $\langle \tilde{\gamma}_0 \rangle_+$ and $\langle \gamma_0 \rangle_+$ are defined by (12), then from (9):

$$\lim_{T \to T_c} \langle \tilde{\gamma}_0^\infty \rangle_+ = \lim_{T \to T_c} \langle \gamma_0^\infty \rangle_+, \tag{44}$$

which from (10) and the second equation of (40) gives

$$\lim_{T \to T_c} R_\uparrow(0^+, T) = 0; \tag{45}$$

so that no jump occurs at T_c.

In this case we can introduce two critical indices:

$$R_\downarrow(p) \sim (p - p_c)^{\beta_p} \simeq (T_c - T)^{\beta_T}, \qquad p_c = \tfrac{1}{2}. \tag{46}$$

On the other side:

$$M(0^+, T) = (p - p_c) \sim (T_c - T)^{\beta}, \qquad \beta = \tfrac{1}{8}. \tag{47}$$

From (38) and (40):

$$\beta_p \leqslant 1; \qquad \beta_T \leqslant \beta.$$

In the same way, from (31) and theorems 1 and 3:

$$\begin{aligned} &\chi(0^+, T) \leqslant 4\langle \gamma_0 \rangle_+ S_\downarrow(0^+, T), && H = 0^+, T < T_c; \\ &\chi(0, T) \leqslant 2S_\downarrow(0, T), && H = 0, T \geqslant T_c. \end{aligned} \tag{48}$$

Since

$$\left.\begin{aligned} &\chi(0^+, T) \sim (T_c - T)^{-\gamma'}, && T < T_c, \\ &\chi(0^+, T) \sim (T - T_c)^{-\gamma}, && T \geqslant T_c, \end{aligned}\right\} \quad \gamma' = \gamma = \tfrac{7}{4}; \tag{49}$$

if we define two other critical indices:

$$\begin{aligned} &S_\downarrow(0^+, T) \sim (T_c - T)^{-\gamma_T'}, && T < T_c, \\ &S_\downarrow(0^+, T) \sim (T - T_c)^{-\gamma_T}, && T \geqslant T_c, \end{aligned} \tag{50}$$

from (48) we have:

$$\gamma' \leqslant \gamma_T'; \qquad \gamma \leqslant \gamma_T. \tag{51}$$

If we suppose that the expression

$$\lim_{T \to T_c} \sum_s s\tilde{n}_{0s}(0^+, T) = \lim_{T \to T_c} \sum_s s n_{0s}(0^+, T), \tag{52}$$

is true, from (30) and (44) it follows that

$$\lim_{T \to T_c} S_\downarrow(0^+, T) = \lim_{T \to T_c} S_\uparrow(0^+, T).$$

Consequently, from (50) and (51) $S_\uparrow(0^+, T)$ diverges at T_c.

Defining

$$S_\uparrow(0^+, T) \sim (T_c - T)^{-\tilde{\gamma}_T'} \qquad T < T_c,$$

from (18) and (30): $\tilde{\gamma}_T' \leqslant \gamma_T'$.

Let us define the connectedness length as

$$\xi_c^{-1}(H, T) = -\lim_{r_{ij} \to \infty} \frac{\ln p_{ij}(H, T)}{r_{ij}};$$

and its critical indices

$$\xi_c(0^+, T) \sim (T_c - T)^{-\nu'_T} \qquad T < T_c$$
$$\xi_c(0, T) \sim (T_c - T)^{-\nu_T} \qquad T \geqslant T_c$$

in analogy with the correlation length (Fisher and Burford 1967)

$$\xi^{-1}(H, T) = -\lim_{r_{ij}\to\infty} \frac{\ln g_{ij}(H, T)}{r_{ij}}$$

and its critical indices

$$\left.\begin{aligned} \xi(0^+, T) &\sim (T_c - T)^{-\nu'}, \qquad T < T_c, \\ \xi(0, T) &\sim (T_c - T)^{-\nu}, \qquad T \geqslant T_c, \end{aligned}\right\} \qquad \nu = \nu' = 1.$$

From (26) and theorem 3 (no infinite (−)-clusters):

$$\nu'_T \geqslant \nu'; \qquad \nu_T \geqslant \nu.$$

In conclusion, for planar models the critical point ($H = 0$, $T = T_c$) is also the critical point for percolation as conjectured by Coniglio (1975) and Odagaki *et al* (1975).

5. Phase diagram of infinite clusters

Theorems 1 and 3 allow us to conjecture the phase diagram of infinite clusters. In figure 1(*a*) in the plane (H/kT, T), the hatched regions correspond to infinite clusters of 'up' spins ($H > 0$) and 'down' spins ($H < 0$). The two curves starting from T_c correspond to critical points for percolation: they tend asymptotically to the limit of a non-interacting percolation point corresponding to a critical density $p_c^0 \geqslant 1/2$ (theorem 1). The width of the region ($T > T_c$) where there are no infinite clusters should tend to zero in the limit case of the triangular lattice for which the critical density $p_c^0 = 1/2$ as shown by Sykes and Essam (1964). For other planar lattices numerical calculations (Shante and Kirkpatrick 1971, Essam 1973, Sykes *et al* 1976b) show $p_c^0 > 1/2$. Along the two

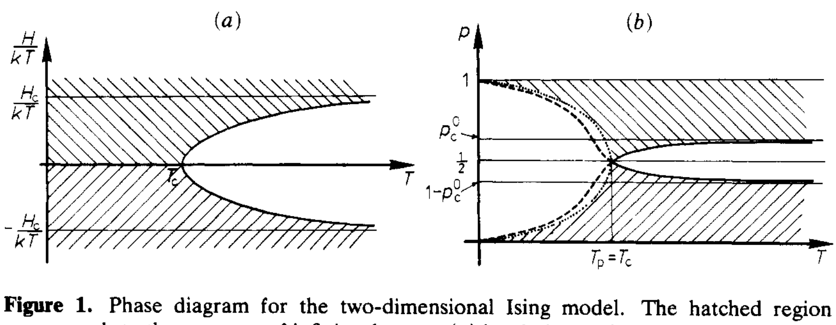

Figure 1. Phase diagram for the two-dimensional Ising model. The hatched region corresponds to the presence of infinite clusters. (*a*) is relative to the temperature reduced magnetic field phase; (*b*) is relative to the temperature–density of 'down' spins phase. The critical point $T = T_c$, $H = 0$ ($p = \frac{1}{2}$) is a percolation point from which two lines of percolation points for 'up' and 'down' spins start. They tend (as $T \to \infty$) to the percolation point for non-interacting spins. In (*b*) the dotted curve is the coexistence curve and the broken curve is the critical line of percolation points.

curves the critical indices should not vary as Monte Carlo calculations show (Odagaki *et al* 1975), but at the critical point they might change. It should be interesting to employ numerical calculation techniques in order to find out if this is true and to discover the relative crossover.

In figure 1(*b*) the same diagram is given in the (p, T) plane. The dotted curve is the coexistence curve. The broken curve corresponds to critical points for percolation in the non-stable region. It is very plausible that they should correspond to spinodals. It would be interesting to investigate this point.

From figure 1(*b*) it is clear that the critical density is an increasing function of the strength of the interaction as this should facilitate the aggregation of clusters. This property has been verified in Monte Carlo calculations (Odagaki *et al* 1975) and is also true in the Bethe lattice (Coniglio 1976) which gives exact results for high values of the dimensionality. We think then that it should be valid also for intermediate dimensionality, $d>2$.

For dimensions $d \geqslant 3$ we can only apply theorems 1 and 2. In fact we should expect that, in this case, given the large connectivity, two infinite clusters of + and − spins can well coexist as happens for large dimensionality (Bethe lattice) (Coniglio 1976), and as numerical calculations show either for non-interacting (Essam 1973) or for interacting spins (Müller-Krumbhaar 1974a).

However we can still introduce some critical indices:

$$R_{\uparrow}(0^+, T) - R_{\downarrow}(0^+, T) \sim (T - T_c)^{\beta_T} \sim (p - p_c)^{\beta_p};$$

and we must still have from theorem 1 that $\beta_T \leqslant \beta$; $\beta_p \leqslant 1$.

The critical point is no longer a percolation point, and one should expect that $S_{\uparrow}(0^+, T)$ and $S_{\downarrow}(0^+, T)$ are finite at T_c. If this is the case from (31) $\chi_{\uparrow}^{\infty}(H, T)$ and $\chi_{\downarrow}^{\infty}(H, T)$ exhibit a divergence stronger than $\chi(H, T)$ at the critical point.

It is instructive to consider the following Hamiltonian:

$$\mathscr{H} = -H \sum_i (\gamma_i^{\infty} - \gamma_i) - H' \sum_i (\tilde{\gamma}_i^{\infty} - \gamma_i^{\infty}) - J \sum_{[ij]} \sigma_i \sigma_j,$$

where we have introduced a field H' which is coupled to the infinite clusters. If $F(H, H', T)$ is the free energy of such a Hamiltonian,

$$kT \frac{\partial F}{\partial H'}\bigg|_{H'=H} = R_{\uparrow}(H, T) - R_{\downarrow}(H, T);$$

$$(kT)^2 \frac{\partial^2 F}{\partial H'^2}\bigg|_{H'=H} = \chi_{\downarrow}^{\infty}(H, T) - \chi_{\uparrow}^{\infty}(H, T);$$

which can be interpreted as a new order parameter and 'susceptibility'.

Figure 2 shows our conjecture of the phase diagram for infinite clusters which is based on theorem 3 and numerical calculations of p_c^0. The hatched region corresponds to infinite clusters. The double hatched region around $p = 1/2$ corresponds to the coexistence of an infinite (+)-cluster with an infinite (−)-cluster. In the figure is also shown T_p, the temperature for percolation at $H = 0$. Note that $T_p < T_c$.

We now want to relate our results to those of the droplet model (Fisher 1967b). This model gives good qualitative results but is unsatisfactory in many respects (Domb 1974) which have been partly removed by modifying it opportunely (Reatto 1970, Reatto and

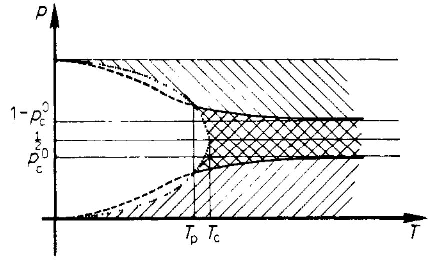

Figure 2. Phase diagram for the three-dimensional Ising model, dividing regions with infinite clusters of 'up' spins from those with infinite clusters of 'down' spins. The double hatched region corresponds to the coexistence of infinite clusters of 'up' and 'down' spins.

Rastelli 1972, Stauffer *et al* 1971). Its peculiarity remains the divergence in the size of the clusters at the critical point. We have shown that this should be the case in two but not in three dimensions. This is, we think, one of the main reasons why this model fits well with the numerical data of Monte Carlo calculations in two dimensions (Stoll *et al* 1972), but not so well in three dimensions (Müller-Krumbhaar 1974b).

We think that another difference between two- and three-dimensional models is based on the shape of finite clusters near the critical point. It is expected that large ramified clusters should be relevant near percolation points, since these will give rise to an infinite cluster with very low density. In two dimensions, as we have shown, critical point and percolation coincide. That is why large ramified clusters should be relevant at the critical point. This is in agreement with the conjecture advanced by Domb (1974) and Stauffer (1975a, b) and also with the suggestion due to Binder (1975). In three dimensions, however, the critical point and the percolation point do not coincide any more and the critical point behaviour will be dominated by an infinite cluster. The compactness depends on how large the percolation probability is for 'down' spins.

6. Conclusions

In conclusion, we believe that from theorems 1, 2 and 3 we have gained a deeper understanding of the phase transitions. In particular the information of the long-range order or long-range correlation can propagate only through infinitely large connected paths of spins in the same state.

From such theorems we have derived phase diagrams for two- and three-dimensional models, which although very plausible have not been derived rigorously. A further study, either numerical or rigorous, to verify these conjectures should be very interesting. The main difference between two- and three-dimensional systems is the different location of the percolation point on the coexistence curve. In the first systems for $H=0$, the percolation temperature T_p coincides with the critical temperature T_c, in the second systems $T_p \leqslant T_c$ (even though there are reasons to believe that for the majority of three-dimensional models $T_p < T_c$). Such a difference between two- and three-dimensional models might be connected to the origin of some of their different properties, such as the behaviour of the surface which separates the plus from the minus phase, which is rigid in three dimensions and fluctuates in two dimensions. These and related problems are under investigation.

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Note added in proof. After this work had been written we received a paper from Sykes and Gaunt (1976), in which series expansion calculations show that for a two-dimensional Ising model $T_p = T_c$ and $\gamma_T = 1{\cdot}91 \pm 0{\cdot}01$ ($\gamma_T > \gamma = 1{\cdot}75$), while for a three-dimensional model $T_p < T_c$. All these results are completely in agreement with our theoretical predictions.

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